Program Description I

Program Title: Summation of an infinite alternating series

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Program Description, Equations, Variables: This program computes the sum of alternating series of the form : $y(0) - y(1) + y(2) - y(3) + \ldots = \sum_{i=0}^{\infty} (-1)^i \, y(i)$ where $i = 0, 1, 2, \ldots$ up to infinity. This is mostly useful when the series converges rather slowly to its limit. For instance, consider the series $S = 1 - 1/2 + 1/3 - 1/4 + \ldots$, whose limit is Ln 2. Were you to find an approximation by actually adding the series terms, you would have to add hundreds of thousands of terms to achieve a mere 4 places approximation.

Of course, there exist more efficient methods to deal with the problem. This program includes 2, the well-known Euler's transformation and the Hutton's transformation:

(1) EULER'S TRANSFORMATION .-

Consider the series

The Euler's transformation replaces $y(0) - y(1) + y(2) - y(3) + \ldots$ by : $= 1/2\, y(0) - 1/4\, \Delta y(0) + 1/8\, \Delta^2 y(0) - \ldots$, where the $\Delta^n y(0)$ are the n-th order forward difference of $y(i)$; up to 17th-order differences may be used.

The program proceeds as follows: given $y(i)$, (which is defined by the user under LBL E, using 41 steps at most , and (TURN PAGE)

Operating Limits and Warnings
- increasing the number of differences and/or previous terms added increases both accuracy and running time.
- some divergent series may be treated too, but the result is not guaranteed in all cases.
- The Euler's transf. is most efficient for very slowly convergent series. The Hutton's one deals equally well with any series.
- $y(i)$ must be strictly positive

using R_A , R_B ,if required) , n terms of the series (n is a user's specified value) are summed in advance, to give S'.Then a difference table is formed thus:

$$\begin{array}{ccccc} y(n+1) & & & & \\ & \Delta y(n+1) & & & \\ y(n+2) & & \Delta^2 y(n+1) & & \\ & \Delta y(n+2) & & \Delta^3 y(n+1) & \ldots \\ y(n+3) & & \ldots & & \\ & \ldots & & & \\ \ldots & & & & \end{array}$$

computing differences up to the m-th order (m is a user's specified value, $1 \le m \le 17$). And finally, the Euler's transformation is applied to yield $S'' = \frac{1}{2}y(n+1) - \frac{1}{4}\Delta y(n+1) + \frac{1}{8}\Delta^2 y(n+1) - \ldots$ and then, $S = S' + S''$ is the final result.Some useful remarks:

a) The ET is most efficient for series whose convergence is very slow, so that y(i) tends to zero like 1/i or so.

b) Tabulation of differences may require a long running time if m is great. In general, $m \le 10$ is advisable. I think that n = m = 8 is a very good election.

2) <u>HUTTON's TRANSFORMATION</u> .- The Hutton's transformation deals directly with the sequence of partial sums S_k, such that:

$S_k = y(0) - y(1) + y(2) - \ldots + (-1)^k\, y(k)$, and replaces the sequence S_0, S_1, S_2,... by a new sequence T_0, T_1, T_2,... where $T_k = \frac{1}{2}(S_k + S_{k-1})$, after which the transformation may be iterated. In fact, the programs sums n terms of the series in advance and then computes and stores S_{n+1}, S_{n+2}, ... , S_{n+9} , after which 9 iterations of Hutton's transformation are performed. Remarks:

a) Hutton's t. applies equally well to either slow or fast convergent series. number of terms to be summed in advance is user's defined; n = 8 is a good value. Accuracy & running time depends on n.

b) The successive approximations afforded by every iteration of Hutton's t. may be printed at will, to see how the method converges. This is stated using the toggle device LBL d.

Program Description II

~~xxxxxxx~~ c) Some considerations: divergent series may be treated as well. If a sum for the divergent series is obtained, it is its Euler's sum (or its Hutton's sum; it both exist, they are equal)

d) A great deal of program instructions have been duplicated, to increase program speed. Also, the iterations of Hutton's t. are straightforward, without branches, to avoid a loop within a loop.

e) Any alternating series may be summed: simply define the general term so that includes all terms of your series. See examples.

Sample Problem(s) (1) Sum the series S = 1-1/2+1/3-1/4+... to 10 places.

- define y(i) = 1/(1+i) : $\boxed{\text{GTO E}}$, switch to W/PRGM, 1 $\boxed{+}$ $\boxed{1/X}$ $\boxed{\text{RTN}}$, switch to RUN
- now, SUM = 10 , DIF = 7: 10 $\boxed{B}$ 7 $\boxed{C}$ $\boxed{A}$ → 0.693147182
- Thus, the Euler's method, after 69 seconds of furious calculation gives the value 0.693147182. The exact value is Ln2 = 0.693147181
- Let's see how does the Hutton's: $\boxed{D}$ → 0.6931471805 so the Hutton's method (after 56 seconds) gives 0.6931471805 ,where Ln2 = 0.6931471806 is the exact result.

(2) FIND THE VALUE OF $S = \frac{1}{2}\int_{K=0}^{K=1}\int_{\theta=0}^{\theta=\pi/2} \frac{d\theta dk}{\sqrt{1-k^2\sin^2\theta}} = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \dots$

- DEFINE $y(i) \Rightarrow 1 - \frac{1}{3^2} + \frac{1}{5^2} - \dots = \sum_{i=0}^{\infty} \frac{1}{(2i+1)^2}$ ⇨ $\boxed{\text{GTO E}}$, SWITCH, 2 $\boxed{X}$ 1 $\boxed{+}$ $\boxed{1/X}$ $\boxed{X^2}$ $\boxed{\text{RTN}}$, SWITCH TO RUN

Solution(s) - LET SUM = 8, DIF = 7 ⇨ 8 $\boxed{B}$ 7 $\boxed{C}$ $\boxed{A}$ → 0.915965595 (70 SECONDS)

TO TEST THE HUTTON'S METHOD $\boxed{D}$ → 0.915965595 (58 SECONDS)

$\boxed{\text{REMARKS}}$ EULER'S T. & HUTTON'S T. ARE INDEPENDENT; IF BOTH GIVE THE SAME RESULT, IT'S ALMOST SURE THAT THE RESULT IS CORRECT. IN THE LAST EXAMPLE, EXACT VALUE IS

$$S = \int_0^1 \frac{\tan^{-1}x}{x}\,dx = 0.915965594$$

(MORE EXAMPLES IN PAGE 4)

Reference(s) - Introduction to numerical analysis - F.B. Hildebrand
International series in pure and applied mathematics - McGraw/Hill
- Chapter 5.9 .- Approximate summation , pages 203 thru 208

(3) FIND THE SUM $S = -\frac{1}{0.23} + \frac{1}{0.24} - \frac{1}{0.25} + \frac{1}{0.26} - \cdots$ **TO 10 PLACES**

- DEFINE GENERAL TERM ⇨ $S = -\left(\frac{1}{0.23} - \frac{1}{0.24} + \cdots\right) = -\sum_{i=0.23}^{\infty} \pm\left(\frac{1}{i}\right) = -\sum_{i=0}^{\infty} (-1)^i \frac{1}{0.23 + 0.01\,i}$ ⇨
  $i = 0.23, 0.24, \ldots$

⇨ $\boxed{GTO\ E}$, SWITCH, $\boxed{RCL\ A}$ $\boxed{X}$ $\boxed{RCL\ B}$ $\boxed{+}$ $\boxed{1/x}$ $\boxed{RTN}$, SWITCH TO RUN

- STORE CONSTANTS ⇨ .01 $\boxed{STO\ A}$ .23 $\boxed{STO\ B}$ 6 $\boxed{C}$ $\boxed{B}$ $\boxed{A}$ → 2.221127525 (55 SECONDS)

SO, EULER'S METHOD PROVIDES $S = -2.221127525$, WHERE THE EXACT RESULT IS $S = -\int_0^1 \frac{1}{x^{0.77}(1+x^{0.01})}dx = -100\int_0^1 \frac{t^{22}}{1+t}dt = -100\left[\ln 2 - \sum_{i=0}^{21} (-1)^i \frac{1}{1+i}\right]$

$= -2.221127525$. TO TEST HUTTON'S METHOD ⇨ 0 $\boxed{B}$ → 0
$\boxed{D}$ → 2.221127524 (41 SEC

(4) FIND THE SUM $1 - (\frac{1}{3})^5 + (\frac{1}{5})^5 - (\frac{1}{7})^5 + \cdots = \sum_0^{\infty} (-1)^i (2i+1)^{-5}$

- THE PREVIOUS EXAMPLES WERE VERY SLOWLY CONVERGENT SERIES. THIS ONE CONVERGES FASTER
- DEFINE GENERAL TERM ⇨ $\boxed{GTO\ E}$, SWITCH, $\boxed{2}$ $\boxed{X}$ 1 $\boxed{+}$ $\boxed{RCL\ A}$ $\boxed{y^x}$ $\boxed{RTN}$, SWITCH TO RUN
- STORE CONSTANTS ⇨ 5 $\boxed{CHS}$ $\boxed{STO\ A}$ 10 $\boxed{B}$ 2 $\boxed{C}$ $\boxed{A}$ → 0.9961578289 (41 SECONDS

EXACT RESULT IS $5\pi^5/1536 =$ 0.9961578288, SO ET BEHAVES VERY WELL.

- LET'S SEE HOW HUTTON'S T. PERFORMS IT'S WORK. SELECT "PRINT" ⇨ $\boxed{fD}$ 1.000000000
  4 $\boxed{B}$ $\boxed{D}$ → 0.996157835 → 0.996157827 → 0.996157829 → 0.996157828 →
  → 0.996157829 → 0.996157829 → 0.996157830 → 0.996157829 →
  → 0.996157829, $\boxed{fD}$ → 0.000000000, "NOT PRINT"
- AS YOU MAY SEE, 3 ITERATION WOULD HAVE SUFFICE, BUT 8 WERE PERFORMED, AS THE NUMBER OF ITERATIONS IS NOT USER'S SELECTABLE.

(5) FIND THE EULER'S SUM OF $S = 1 - 1 + 1 - 1 + 1 - \cdots$

- THIS SERIES IS NEITHER CONVERGENT NOR DIVERGENT, AS ITS PARTIAL SUMS ARE 0, 2,
- DEFINE GENERAL TERM ⇨ $\boxed{GTO\ E}$, SWITCH, 1 $\boxed{RTN}$, SWITCH TO RUN
- SUM = 0, DIF = 1 ⇨ 0 $\boxed{B}$ 1 $\boxed{C}$ $\boxed{A}$ → 0.5000000000 (8 SECONDS)

THE EXACT RESULT IS ⇨ $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots = y(x)$
$y(1) = 1 - 1 + 1 - 1 + \cdots = \frac{1}{1+1} = \frac{1}{2} = 0.5$

- WHAT DOES HUTTON'S ? ⇨ $\boxed{D}$ → 0.5000000000 (38 SECONDS)

(6) FIND THE EULER'S (HUTTON'S) SUM OF $1 - 2 + 3 - 4 + 5 - 6 + \cdots$

- THIS IS A CLEARLY DIVERGENT SERIES
- DEFINE GENERAL TERM ⇨ $1 - 2 + 3 - \cdots = \sum_{i=0}^{\infty} (-1)^i (1+i)$ ⇨ $\boxed{GTO\ E}$, SWITCH, 1 $\boxed{+}$ $\boxed{RTN}$, S
- SUM = 0, DIF = 1 ⇨ 0 $\boxed{B}$ 1 $\boxed{C}$ $\boxed{A}$ → 0.2500000000 (9 SECONDS)

THE EXACT RESULT IS ⇨ $\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots = y(x)$ ⇨ $y(1) = 1 - 2 + 3 - \cdots = \frac{1}{(1+1)^2}$

- THE HUTTON'S SUM ⇨ $\boxed{D}$ → 0.2500000000 (39 SECONDS)

User Instructions

SUMMATION OF INIFINITE ALTERNATING SERIES				
			PRINT ?	
EULER	# SUM	# DIF	HUTTON	i → y(i)

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS		OUTPUT DATA/UNITS
1	LOAD PROGRAM , BOTH SIDES				
2	DEFINE GENERAL TERM y(i) ⇨ PRESS		GTO	E	
	SWITCH TO W/PRGM , AND ENTER THE SEQUENCE				
	OF KEYSTROKES THAT CALCULATES y(i) , WHERE i				
	IS IN THE DISPLAY . THEN PRESS		RTN		
	AND SWITCH TO RUN . y(i) MAY USE UP TO 40				
	STEPS , AND REGISTERS RA, RB (IF EULER'S METHOD IS				
	TO BE USED) OR RA, B, D, E, 0, 1, ..., 7 (HUTTON'S)				
3	TO USE EULER'S TRANSFORMATION				
3a	$(SUM \geq 0)$ - INPUT NUMBER OF PREVIOUS TERMS ADDED	SUM	B		SUM
3b	$(1 \leq DIF \leq 17)$ - INPUT NUMBER OF DIFFERENCES USED	DIF	C		DIF
	- COMPUTE AN APPROXIMATION TO THE SUM		A		$\sum_0^{\infty}(-1)^i y(i)$
4	FOR ANOTHER VALUES OF "SUM" OR "DIF", GOTO 3a or 3b				
5	FOR ANOTHER CASE, GO TO 2				
6	TO USE HUTTON'S TRANSFORMATION				
6a	$(SUM \geq 0)$ - INPUT NUMBER OF PREVIOUS TERMS ADDED	SUM	B		SUM
	- SELECT "PRINT" OR "NOT PRINT"		f	D	1.00 ...
	(DEFAULT = "NOT PRINT" = 0.00 ...)		f	D	0.00 ...
	- COMPUTE AN APPROXIMATION TO THE SUM		D		$\sum_0^{\infty}(-1)^i y(i)$
	(IF "PRINT" WAS SELECTED , SUCCESSIVE APPROXI				
	MATIONS , 8 IN TOTAL, WILL BE PRINTED)				
7	FOR ANOTHER VALUES OF "SUM" GOTO 6a				
8	FOR ANOTHER CASE , GO TO 2				
9	TO CALCULATE y(i) FOR SOME i	i	E		y(i)
	NOTE : - "SUM" & "DIF" REMAIN STORED				
	THROUGHOUT CALCULATIONS				
	- AFTER COMPUTING EULER'S SUM ,				
	HUTTON'S SUM MAY BE ASKED FOR, OR VICE-				
	-VERSA .				
	- "SUM" & "DIF" ARE POSITIVE INTEGERS				

STEP	KEY ENTRY	KEY CODE	COMMENTS
001	LBL d	32 25 14	"PRINT" TOGGLE
	F ? 1	35 71 01	
	GTO 3	22 03	
	SF 1	35 51 01	SET "PRINT"
	1	01	
	RTN	35 22	
	LBL 3	31 25 03	AUXILIAR LABEL
	CF 1	35 61 01	SET "NOT PRINT"
	0	00	
010	RTN	35 22	
	LBL B	31 25 12	INPUT "SUM"
	STO C	33 13	STORES # OF PREV. SUM.
	RTN	35 22	
	LBL C	31 25 13	INPUT "DIF"
	STO E	33 15	STORES # OF DIFF.
	RTN	35 22	
	LBL D	31 25 14	HUTTON'S TRANSFORMATION
	SF 0	35 51 00	
	GTO 3	22 03	
020	LBL A	31 25 11	EULER'S TRANSFORMATION
	CF 0	35 61 00	
	P⇄S	31 42	
	CF 2	35 61 02	
	LBL 3	31 25 03	AUXILIAR LABEL
	1	01	
	STO 8	33 08	
	0	00	CLEARS S
	STO 9	33 09	
	STO I	35 33	COUNTER RESET TO 0
030	LBL 1	31 25 01	LOOP, COMPUTES $\sum (-1)^i y(i)$
	GSB E	31 22 15	
	RCL 8	34 08	$y(i) \times \left\{ \begin{matrix} 1 \\ -1 \end{matrix} \right.$
	STO- 8	33 51 08	
	STO- 8	33 51 08	
	x	71	
	STO + 9	33 61 09	COMPUTES $S' = \sum_0^{SUM} (-1)^i y(i)$
	ISZ	31 34	ANOTHER TERM?
	RCL C	34 13	
	RCL I	35 34	YES, LOOP
040	x ≤ y	32 71	
	GTO 1	22 01	
	F ? 0	35 71 00	HUTTON?
	GTO 5	22 05	YES
	STO 8	33 08	NO, EULER
	2	02	ODD OR EVEN?
	÷	81	
	FRAC	32 83	
	x ≠ 0	31 61	
	SF 2	35 51 02	
050	CLX	44	RESET COUNTER TO 0
	STO I	35 33	
	LBL 2	31 25 02	LOOP, TABLE OF DIFFERENCES
	RCL 8	34 08	
	GSB E	31 22 15	
	P⇄S	31 42	COMPUTES AND STORES y(m+1), y(m+2), ...
	STO (i)	33 24	
	P⇄S	31 42	
	ISZ	31 34	
	1	01	ANOTHER TERM?
060	STO + 8	33 61 08	
	RCL E	34 15	
	RCL I	35 34	
	x ≤ y	32 71	
	GTO 2	22 02	YES, LOOP.
	2	02	
	F ? 2	35 71 02	ODD?
	CHS	42	YES, −2 IN R_8
	STO 8	33 08	
	ABS	35 64	
070	LBL 6	31 25 06	DIFFERENCES SCHEME
	−	51	2 LOOPS (NESTED)
	STO I	35 33	
	STO D	33 14	
	LBL 0	31 25 00	MAIN LOOP
	P⇄S	31 42	COMPUTES EACH DIFFERENCE REQUIRED
	RCL (i)	34 24	
	ISZ	31 34	
	STO− (i)	33 51 24	
	P⇄S	31 42	
080	RCL E	34 15	1ST LOOP CONTROL
	RCL I	35 34	
	x ≠ y	32 61	
	GTO 0	22 00	
	RCL D	34 14	2ND LOOP CONTROL
	STO I	35 33	
	x = 0	31 51	
	GTO 4	22 04	
	1	01	
	GTO 6	22 06	
090	LBL 4	31 25 04	EULER'S TRANSFORMATION
	P⇄S	31 42	$S = S' + S''$
	RCL (i)	34 24	
	P⇄S	31 42	
	RCL 8	34 08	WHERE $S' = \sum_0^{SUM} (-1)^i$
	÷	81	
	STO + 9	33 61 09	$S'' = \frac{1}{2} y(m+1) -$
	2	02	
	CHS	42	$- \frac{1}{4} \Delta y(m+1) + \frac{1}{8} \Delta^2 y(m$
	STO x 8	33 71 08	
100	ISZ	31 34	$- \dots$
	RCL E	34 15	
	RCL I	35 34	
	x ≤ y	32 71	
	GTO 4	22 04	
	GTO 9	22 09	OUTPUT $\sum_0^{\infty} (-1)^i y(i)$
	LBL 5	31 25 05	HUTTON'S
	9	09	
	STO I	35 33	
	LBL 7	31 25 07	PARTIAL SUMS LOOP
110	RCL C	34 13	
	1	01	
	0	00	

REGISTERS

0	1	2	3	4	5	6	7	8	9
y(m+1)	Δy(m+1)	Δ^2y(m+1)	Δ^3y(m+1)	Δ^4y(m+1)	Δ^5y(m+1)	Δ^6y(m+1)	Δ^7y(m+1)	Δ^8y(m+1)	Δ^9y(m+1)

S0	S1	S2	S3	S4	S5	S6	S7	S8	S9
Δ^{10}y(m+1) / Sm+0	Δ^{11}y(m+1) / Sm+1	Δ^{12}y(m+1) / Sm+2	Δ^{13}y(m+1) / Sm+3	Δ^{14}y(m+1) / Sm+4	Δ^{15}y(m+1) / Sm+5	Δ^{16}y(m+1) / Sm+6	Δ^{17}y(m+1) / Sm+7	+1 OR −1, etc / Sm+8	Σ / Sm+

A	B	C	D	E	I
		m = SUM	used	m = DIF	used.

STEP	KEY ENTRY	KEY CODE	COMMENTS
	+	61	
	RCL I	35 34	
	-	51	COMPUTES AND STORES
	GSB E	31 22 15	IN RS1, RS2, --
	RCL 8	34 08	
	STO- 8	33 51 08	-- RS9
	STO- 8	33 51 08	
	x	71	THE PARTIAL SUMS
	STO+ 9	33 61 09	
	RCL 9	34 09	$S_{m+1}, S_{m+2}, \dots S_{m+9}$
	P⇄S	31 42	
	STO (i)	33 24	
	P⇄S	31 42	
	DSZ	31 33	LOOP CONTROL
	GTO 7	22 07	
	8	08	
	STO I	35 33	
130	P⇄S	31 42	
	LBL 8	31 25 08	HUTTON'S TRANSFORMATION
	RCL 8	34 08	
	STO+ 9	33 61 09	
	2	02	
	STO: 9	33 81 09	STARTING FROM A
	RCL 7	34 07	PREVIOUS ITERATION
	STO+ 8	33 61 08	
	2	02	, IT COMPUTES A NEW
	STO: 8	33 81 08	
140	RCL 6	34 06	SEQUENCE OF PARTIAL
	STO+ 7	33 61 07	SUMS , MAKING USE
	2	02	
	STO: 7	33 81 07	OF
	RCL 5	34 05	
	STO+ 6	33 61 06	$T_K = \frac{1}{2}(S_K + S_{K-1})$
	2	02	
	STO: 6	33 81 06	
	RCL 4	34 04	WHERE THE S_K
	STO+ 5	33 61 05	
150	2	02	ARE STORED IN
	STO: 5	33 81 05	
	RCL 3	34 03	RS1 THRU RS9
	STO+ 4	33 61 04	
	2	02	
	STO: 4	33 81 04	
	RCL 2	34 02	
	STO+ 3	33 61 03	
	2	02	
	STO: 3	33 81 03	
160	RCL 1	34 01	
	STO+ 2	33 61 02	
	2	02	
	STO: 2	33 81 02	
164	F ? 1	35 71 01	"PRINT"?
165	GSB 3	31 22 03	YES
	DSZ	31 33	ITERATIONS CONTROL
	GTO 8	22 08	
	LBL 9	31 25 09	AUXILIAR LABEL

STEP	KEY ENTRY	KEY CODE	COMMENTS
	RCL 9	34 09	$\sum_{i=0}^{\infty}(-1)^i \; y(i)$
170	P⇄S	31 42	
	RTN	35 22	
172	LBL 3	31 25 03	"PRINT" LABEL
	1	01	
	0	00	
	RCL I	35 34	
	-	51	
	STO I	35 33	
	RCL (i)	34 24	PRINTS EACH SUCCE[SSIVE]
	-x-	31 84	APPROXIMATION
180	LAST X	35 82	
	STO I	35 33	
182	RTN	35 22	
	LBL E	31 25 15	y(i) EVALUATION
190			
200			
210			
220			

NOTE

IF PRINT OF APPROX[I]MATIONS ISN'T NEEDED OR WANTED, THE FOLLOW[ING] STEPS MAY BE DELETED.

001 THROUGH 010
164 AND 165
172 THROUGH 182
(ALL LIMITS INCLUDED)

THIS GIVES 23 ADDITIONAL STEPS FOR DEFINITION OF y(i) OR OTHERWISE.

LABELS

A	B	C	D	E
EULER	SUM	DIF	HUTTON	i → y(i)
a	b	c	d PRINT ?	e
0 used	1 used	2 used	3 used	4 used
5 used	6 used	7 used	8 used	9 used

FLAGS

0	HUTTON ?
1	PRINT ?
2	ODD / EVEN
3	

SET STATUS

FLAGS	ON	OFF	TRIG		DISP	
0	☐	■	DEG	☐	FIX	■
1	☐	■	GRAD	☐	SCI	☐
2	☐	■	RAD	■	ENG	☐
3	☐	■			n	9