Program Description I

Program Title =AREAS, LENGTH OF ARCS, VOLUMES & SURFACES OF REVOLUTION GENERATED BY CURVES EXPRESSED EITHER IN RECTANGULAR OR POLAR COORDINATES=

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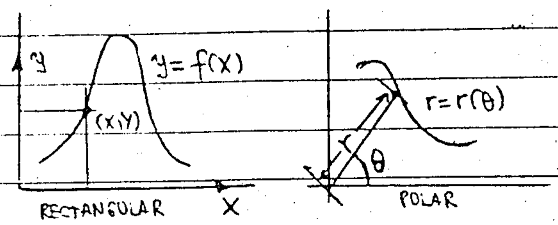
Program Description, Equations, Variables: GIVEN ANY USER'S DEFINED FUNCTION THAT REPRESENTS THE EQUATION OF A 2-DIMENSIONAL CURVE, EITHER IN RECTANGULAR OR POLAR COORDINATES, THE PROGRAM FINDS THE AREA BOUNDED BY THE CURVE, THE LENGTH OF THE ARC, AND THE VOLUME AND AREA OF THE SOLID OF REVOLUTION GENERATED BY THE CURVE WHEN REVOLVING AROUND THE X AXIS



TO SEE ALL NECESSARY DETAILS, GO TO PAGE 2. ALL OF THESE COMPUTATIONS REQUIRE AN INTEGRATION ROUTINE, AND SOME OF THEM, A DERIVATION ROUTINE.

THE FOLLOWING FORMULAS ARE USED:

$\boxed{\text{DERIVATION}}$

$$y'(x) \simeq \frac{y(x+\Delta x)-y(x-\Delta x)}{2\Delta x}$$

, WHERE Δx IS ARBITRARILY SET TO 8.10^{-4}. THE RESULT REMAINS STORED IN R3

= THIS IS ONLY AN APPROXIMATION (A RATHER GOOD ONE, INDEED), SO ALL CALCULATIONS THAT USE IT AS A SUBROUTINE WILL HAVE SOME ERROR (APART FROM ROUNDING ERRORS).

$\boxed{\text{INTEGRATION}}$

$\int_a^b f(x)\,dx \Rightarrow$ THE CHANGE OF VARIABLE $\left\} \; x=\frac{b+a}{2}+\frac{b-a}{2}t, \quad dx=\frac{b-a}{2}dt \right.$

TRANSFORMS THE $[a,b]$ INTERVAL INTO $[-1,1]$. THE RESULTING INTEGRAL IS THEN

~~Operating Limits and Warnings~~ APPROXIMATED BY:

$$\int_{-1}^{1} y(x)\,dx \simeq \frac{5}{6}\left[y\left(\sqrt{\tfrac{1}{5}}\right)+y\left(-\sqrt{\tfrac{1}{5}}\right)\right]+\frac{1}{6}\left[y(1)+y(-1)\right]+\text{Error}$$

WHERE THE ERROR TERM IS ZERO IF $y(x)$ IS A POLYNOMIAL OF DEGREE 5 OR LESS (BY THE WAY, THE WELL-KNOWN SIMPSON'S RULE IS EXACT ONLY FOR $y(x) \equiv$ POLYNOMIAL OF DEGREE 3 OR LESS)

=THE INTERVAL $[-1,1]$ MAY BE DIVIDED INTO m SUBINTERVALS (WHERE m IS CHOSEN BY THE USER) AND THE PRECEDING FORMULA IS APPLIED TO EACH ONE, TO INCREASE ACCURACY AS MUCH AS DESIRED.

This program has been verified only with respect to the numerical example given in *Program Description II*. User accepts and uses this program material AT HIS OWN RISK, in reliance solely upon his own inspection of the program material and without reliance upon any representation or description concerning the program material.

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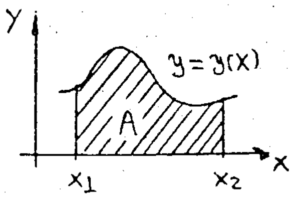
LBL A ≡ AREA BOUNDED BY A CURVE $y = y(X)$ OR $\int_{X_1}^{X_2} y(X)\,dx$



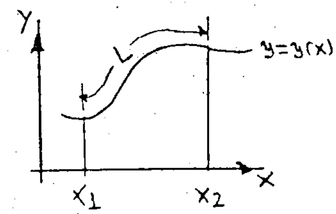
– IT COMPUTES THE AREA A BOUNDED BY A CURVE $y = y(X)$, THE X AXIS, AND 2 ARBITRARY LIMITS X_1, X_2, USING THE FORMULA:

$$A = \int_{X_1}^{X2} y(x)\,dx$$

{ THE EQUATION OF THE CURVE $y = y(X)$ IS GIVEN IN (CA) RECTANGULAR COORDINATES X

– IT ALSO COMPUTES THE INTEGRAL OF SOME USER'S DEFINED FUNCTION $y = y(X)$ OVER THE INTERV $[X_1, X_2]$, USING THE SAME FORMULA. THE NUMBER OF SUBINTERVALS IS ARBITRARILY CHOSEN IF $y(X)$ IS A POLYNOMIAL OF DEGREE 5 OR LESS, THE RESULT IS EXACT REGARDLESS OF X_1, X_2, 1 SUBINTERVAL WILL DO. THE INTEGRAL REMAINS ALWAYS STORED IN R_8

LBL B ≡ LENGTH OF ARC OF A CURVE $y = y(X)$

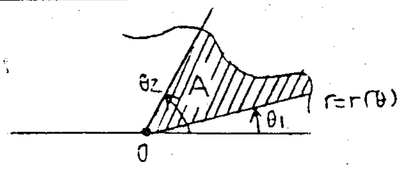


– IT COMPUTES THE LENGTH OF ARC OF THE CURVE $y = y(X)$ (GIVEN IN RECTANGU COORDINATES X, Y), BETWEEN THE ARBITRARY LIMITS X_1, X_2, USING THE FORMULA

$$L = \int_{X_1}^{X_2} \sqrt{1 + [y'(x)]^2}\; dx$$

$y(X)$ & $y'(X)$ MUST BE CONTINUOUS FUNCTION ON $[X_1, X_2]$. $y'(X)$ MUST NOT BE INF

LBL C ≡ AREA BOUNDED BY A CURVE $r = r(\theta)$

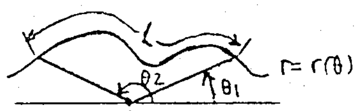


– IT COMPUTES THE AREA A BOUNDED BY A CURVE $r = r(\theta$ EXPRESSED IN POLAR COORDINATES r, θ, AND 2 ARBITRARY ANGLES θ_1, θ_2, USI THE FORMULA

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} [r(\theta)]^2\, d\theta$$

BY THE WAY, ALL INTEGRALS MAY BE CALCULATED WITH m SUBINTERVALS.

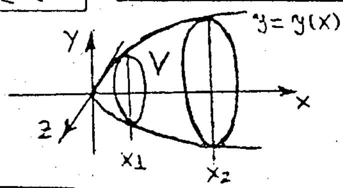
LBL D ≡ LENGTH OF ARC OF A CURVE $r = r(\theta)$



– IT COMPUTES THE LENGTH OF ARC OF THE CURVE $r = r(\theta)$, EXPRESSED IN POLAR COORDI r, θ, BETWEEN 2 ARBITRARY ANGLES θ_1, θ_2, USING THE FORMULA:

$$L = \int_{\theta_1}^{\theta_2} \sqrt{[r(\theta)]^2 + [r'(\theta)]^2}\; d\theta \quad , \quad r' \begin{cases} \text{CONTINUOUS} \\ \text{NON-INFINITE} \end{cases}$$

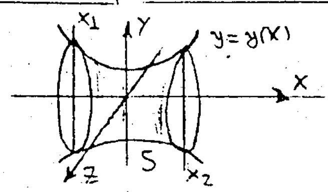
LBL a ≡ VOLUME OF A SOLID OF REVOLUTION



– IT COMPUTES THE VOLUMEN (BETWEEN 2 ARBITRARY LIMITS X_1, X_2) OF REVOLU GENERATED BY A CURVE $y = y(X)$ EXPRESSED IN RECTANGULAR COORDINATES X WHEN REVOLVING AROUND THE X AXIS, USING THE FORMULA

$$V = \pi \int_{X_1}^{X_2} [y(x)]^2\, dx$$

LBL b ≡ AREA OF A SURFACE OF REVOLUTION



– IT COMPUTES THE AREA OF THE SURFACE (BETWEEN X_1, X_2) OF REVOLUTION GENERATED A CURVE $y = y(X)$ EXPRESSED IN RECTANGULAR COORDINATES X, Y, WHEN REVOLVING AROUND THE X AXIS, USING THE FORMULA

$$S = 2\pi \int_{X_1}^{X_2} y(x) \cdot \sqrt{1 + [y'(x)]^2}\; dx \quad , \quad y' = \begin{cases} \text{CONTINUOUS} \\ \text{NON-INFINI} \end{cases}$$

LBL c ≡ STORES THE # OF SUBINTERVALS, m, THAT ARE USED IN THE INTEGRATION ROUTINE. m MUST BE POS AND INTEGER, AND IS STORED IN R_0. LARGE VALUES OF m MAY RESULT IN A EXTREMELY LONG RUNNING

LBL e ≡ COMPUTES THE DERIVATIVE OF $y(X) \Rightarrow y'(X) \simeq \dfrac{y(X+\Delta) - y(X-\Delta)}{2\Delta}$. SHOWS Error IF EITHER $y(X+\Delta)$ OR $y(X-\Delta)$ IS NOT DEFINED. ACCURACY IS ABOUT 6 OR 7 SIGNIFICANT FIGURES

LBL d ≡ AUTOMATIC EVALUATION ; IT IS VERY USEFUL TO PLOT $y = y(X)$ OR $r = r(\theta)$. SIMPLY, PRESS

X_0 $\boxed{\text{ENTER}}$ Δ $\boxed{f\ D}$ $\rightarrow (X_0) \rightarrow y(X_0)$
$\rightarrow (X_1) \rightarrow y(X_1) \rightarrow$ etc, ... } WHERE $X_{i+1} = X_i +$

– THE SAME APPLIES TO $r = r(\theta)$. TO STOP THE AUTOMATIC EVALUATION, PRESS $\boxed{R/S}$

LBL E ≡ $y(X)$ OR $r(\theta)$ DEFINITION & EVALUATION (R_{S0} TO R_{S9} MAY BE USED TO DEFINE THE CURVE)

– TO DEFINE A FUNCTION (OR A CURVE) $y = y(X)$ OR $r = r(\theta)$, PRESS $\boxed{\text{GTO E}}$, SWITCH PRGM, AND INTRODUCE THE SEQUENCE OF KEYSTROKES THAT CALCULATES $y(X)$ OR $r(\theta)$, WHERE X (OR θ IN THE DISPLAY AT THE BEGINNING. PRESS $\boxed{\text{RTN}}$, AND SWITCH TO RUN.

TO EVALUATE $y(X)$ (OR $r(\theta)$) AT SOME POINT X (OR θ) $\Rightarrow$ X (OR θ) $\boxed{E} \rightarrow y(X)$ [OR $r(\theta$

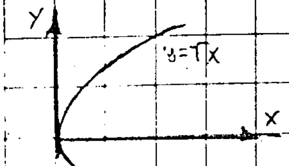
Program Description II

(Sketch(es)) WARNING

≡ INCREASING N INCREASES ACCURACY AND RUNNING TIME.

≡ IF AN Error DISPLAY DOES SHOW UP, CHECK THAT $y'(x)$ IS A FINITE VALUE IN EVERY POINT OF THE INTEGRATION INTERVAL.

FOR INSTANCE, THE DERIVATIVE OF $y = \sqrt{x}$ CANNOT BE EVALUATED AT $x=0$, BECAUSE

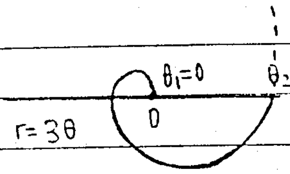


$y' = \frac{1}{2\sqrt{x}}$, AND $y'(0) \to \infty$. IF YOU TRIED TO USE THE DERIVATION ROUTINE, AN Error MESSAGE WOULD BE OBTAINED, SINCE $y(x-\Delta) = y(0 - 8.10^{-4}) = y(-8.10^{-4}) = \sqrt{-8.10^{-4}}$, WHICH RESULTS IN Error

Sample Problem(s) (1) FIND THE LENGTH OF THE 1ST TURN OF THE SPIRAL OF ARCHIMEDES DEFINED BY ITS POLAR EQUATION $r = 3\theta$

– LOAD PROGRAM

– DEFINE $r(\theta)$ ⇨ [GTO E], SWITCH TO PRGM, [3] [X] [RTN], SWITCH TO RUN



– USING 2 SUBINTERVALS ⇨ 2 [f C] → 2.0000

– ENTER LIMITS ⇨ 0 [ENTER] 2π [D] → 63.7675

– TO CHECK RESULT (AND INCREASE ACCURACY), WE WILL USE 8 SUBINTERVALS.

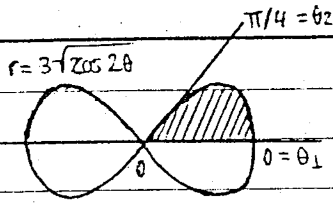
8 [f C] [DSP 6] 0 [ENTER] 2π [D] → 63.768879

– THIS IS, $L = 63.768879+$. EXACT RESULT IS $L = 3\pi\sqrt{1+4\pi^2} + \frac{3}{2}\operatorname{Ln}(2\pi + \sqrt{1+4\pi^2}) =$ $= 63.768882+$, SO Error $\simeq 3.10^{-6}$, QUITE SMALL

(2) FIND THE AREA BOUNDED BY THE LEMNISCATE DEFINED BY $r = 3\sqrt{\cos 2\theta}$

– DUE TO SYMMETRY, THE TOTAL AREA IS 4 TIMES THE AREA OF THE HALF FOIL BETWEEN 0 AND $\pi/4$

Solution(s)



– DEFINE $r(\theta)$ ⇨ [GTO E], SWITCH TO PRGM, [2] [X] [COS] [ABS] [√] [3] [X] [RTN], SWITCH TO RUN

– USING 2 SUBINT ⇨ 2 [f C] 0 [ENTER] π/4 [C] → 2.250000

– SO, THE TOTAL AREA BOUNDED BY THE LEMNISCATE IS $4 \times 2.25 = 9$ (EXACT RESULT IS 9)

(3) FIND THE SUM OF THE SERIES $S = 1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \frac{1}{17} - \dots = \int_0^1 \frac{1}{1+x^4}\,dx$

– DEFINE $y(x)$ ⇨ [GTO E], SWITCH, [x²] [x²] [1] [+] [1/x] [RTN], SWITCH TO RUN, [DSP 8]

– SELECT 4 SUBINT ⇨ 4 [f C] 0 [ENTER] 1 [A] → 0.86697299

– EXACT RESULT IS $S = \frac{\pi\sqrt{2}}{8} + \frac{\sqrt{2}}{4}\operatorname{Ln}(1+\sqrt{2}) = 0.86697299+$

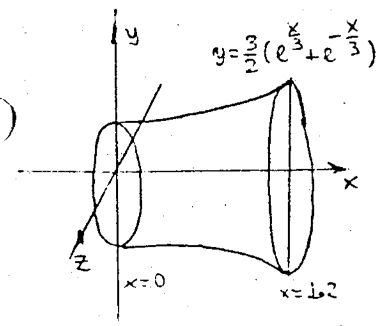
$$= \int_0^1 \frac{dx}{1+x^4}$$

Reference(s)

– DIFERENTSIALNOYE I INTEGRALNOYE ISCHISLENIE – N. PISKUNOV
MEZHKNIGA, MOSCOW (1966)

– NUMERICAL ANALYSIS – F.B. SCHEID
MAC GRAW-HILL (SCHAUM'S OUTLINE SERIES)

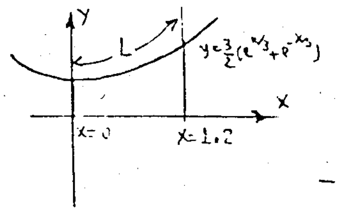
(4) FIND THE VOLUME OF THE SOLID OF REVOLUTION OBTAINED BY THE TURN OF THE CATENARY $y = \frac{3}{2}(e^{x/3} + e^{-x/3})$ AROUND THE X AXIS, BET. X=0, X=1.2



- DEFINE y(X) ⇨ $\boxed{GTO\ E}$, SWITCH, 3 $\boxed{\div}$ $\boxed{e^x}$ $\boxed{ENTER}$ $\boxed{1/x}$ $\boxed{+}$ 1.5 $\boxed{X}$ $\boxed{RTN}$, SW[ITCH] TO RUN.
- USING 2 SUBINTERVALS ⇨ 2 $\boxed{fC}$ $\boxed{DSP9}$ 0 $\boxed{ENTER}$ 1.2 $\boxed{fA}$ → $\underline{35.79755419}$
- THE EXACT RESULT IS $\frac{2\pi}{16} \cdot 27\left(e^{0.8} - e^{-0.8}\right) + 0.6\,[9\pi] =$

$= \underline{35.79755412+}$

(5) FIND THE LENGTH OF THE ARC OF THE SAME CATENARY OF THE PREVIOUS EXAMPLE BET. X=0, X=1.[2]

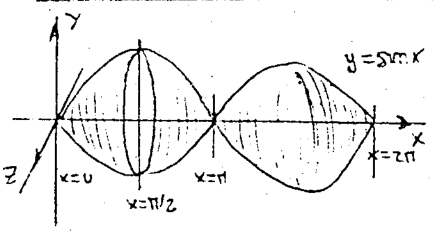


- AS $y(X) = \frac{3}{2}(e^{x/3} + e^{-x/3})$ IS ALREADY DEFINED, AND THE NUMBER (2) OF SUBINTERV. IS ALSO STORED, SIMPLY:

$\boxed{DSP7}$ 0 $\boxed{ENTER}$ 1.2 $\boxed{B}$ → $\underline{1.2322571}$

- exact RESULT IS $\sqrt{[y(1.2)]^2 - 9}$ ⇨ 1.2 $\boxed{E}$ → 3.2432171 $\boxed{x^2}$ 9 $\boxed{-}$ $\boxed{\sqrt{\ }}$ →
→ $\underline{1.2322570+}$

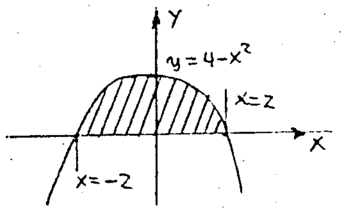
(6) FIND THE AREA OF THE SURFACE OF REVOLUTION CREATED BY $y = \sin(x)$, WHEN ITS ARC FROM X = 0 TO X = 2π TURNS AROUND THE X AXIS.



- DUE TO SYMMETRY, THE AREA IS 4 TIMES THE AREA OF THE SURFACE BETWEEN X=0 AND X=π/2. PRESS $\boxed{DSP6}$
- USING 4 SUBINTERVALS ⇨ DEFINE y(X) ⇨ $\boxed{GTO\ E}$, SWITCH, $\boxed{SIN}$ $\boxed{RTN}$, SWITCH TO PRGM, 4 $\boxed{fC}$ 0 $\boxed{ENTER}$ π/2 $\boxed{fB}$ → $\underline{7.211799}$
- TO OBTAIN THE TOTAL AREA ⇨ 4 $\boxed{X}$ → $\underline{28.847197}$

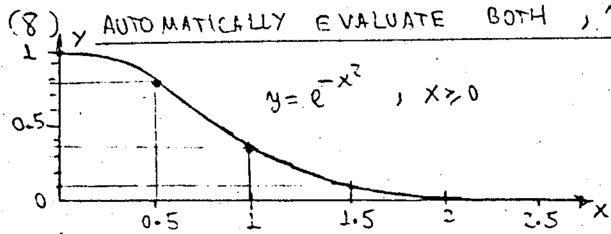
- EXACT RESULT IS $4\pi[\sqrt{2} + \ln(\sqrt{2}+1)] = \underline{28.847199+}$

(7) FIND THE AREA BOUNDED BY $y = 4 - x^2$ AND THE X AXIS

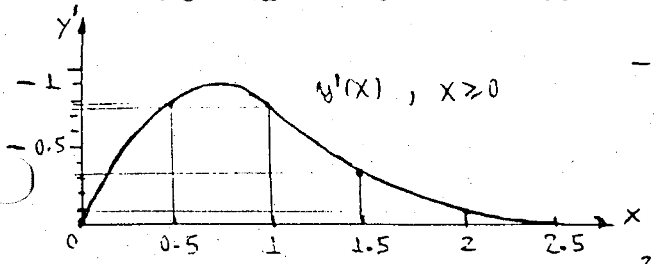


- DEFINE y(X) ⇨ $\boxed{GTO\ E}$, SWITCH, $\boxed{x^2}$ 4 $\boxed{-}$ $\boxed{CHS}$ $\boxed{RTN}$, SWITCH TO RUN
- AS y(X) IS A POLYNOMIAL OF DEGREE 2 ≤ 5, RESULT WILL BE EXACT
- USING 1 SUBINTERVAL ⇨ $\boxed{DSP9}$ 1 $\boxed{fC}$ −2 $\boxed{ENTER}$ 2 $\boxed{A}$ → $\underline{10.66666667}$
- EXACT RESULT IS $10\frac{2}{3} = \underline{10.66666667+}$

(8) AUTOMATICALLY EVALUATE BOTH, $y = e^{-x^2}$ AND ITS 1ST DERIVATIVE, FOR X ≥ 0



- DEFINE y(X) ⇨ $\boxed{GTO\ E}$, SWITCH, $\boxed{x^2}$ $\boxed{CHS}$ $\boxed{e^x}$ $\boxed{RTN}$, SWITCH TO [RUN]
- PRESS $\boxed{DSP4}$ 0 $\boxed{ENTER}$ .5 $\boxed{fD}$ → 0.0000 (X_0) → 1.0000 (y_0)
→ 0.5000 (X_1) → 0.7788 (y_1)
→ 1.0000 (X_2) → 0.3679 (y_2)
→ 1.5000 (X_3) → 0.1054 (y_3)
$\boxed{R/S}$



- TO EVALUATE AUTOMATICALLY y'(X), PRESS ⇨ $\boxed{GTO\ .154}$, SWITCH TO PRGM, $\boxed{DEL}$ $\boxed{f\ E}$, SWITCH TO RUN.

0 $\boxed{ENTER}$ .5 $\boxed{fD}$ → 0.0000 (X_0) → 0.0000 (y'_0)
→ 0.5000 (X_1) → −0.7788 (y'_1)
→ 1.0000 (X_2) → −0.7358 (y'_2)
→ 1.5000 (X_3) → −0.3162 (y'_3)
→ 2.0000 (X_4) → −0.0733 (y'_4)

- THE EXACT $y'(X) = -2x\,e^{-x^2}$

- TO COMPUTE THE DERIVATIVE AT X = 1
$\boxed{DSP7}$ 1 $\boxed{fE}$ → $\underline{-0.7357588}$. THE EXACT VALUE $= -\frac{2}{e} = \underline{-0.7357589+}$

User Instructions

AREAS, LENGTH OF ARCS, VOLUMES OF REVOLUTION					
REVOLUTION: $X_1 \uparrow X_2 \to V$	$X_1 \uparrow X_2 \to S$	# OF SUBINTV.	$X_0 \uparrow \Delta \to (X_i) \to y_i$	$X \to y'(X)$	
RECTANGULAR: $X_1 \uparrow X_2 \to A$	$X_1 \uparrow X_2 \to L$	POLAR: $\theta_1 \uparrow \theta_2 \to A$	$\theta_1 \uparrow \theta_2 \to L$	$X \to y(X)$	$\theta \to r(\theta)$

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS		OUTPUT DATA/UNITS
1	LOAD PROGRAM, BOTH SIDES				
2	DEFINE $F(X) \equiv y(X)$ OR $r(\theta)$ ⇨ PRESS		GTO	E	
	AND SWITCH TO PRGM. INTRODUCE THE SEQUENCE OF KEYS-				
	TROKES THAT CALCULATES $y(X)$ or $r(\theta)$; X or θ MAY BE				
	FOUND IN THE DISPLAY. AT THE END OF THE SEQUENCE				
	INSERT $\boxed{RTN}$. 63 STEPS & RS0 THROUGH RS9 MAY BE USED.				
3	TO EVALUATE $y(X)$ OR $r(\theta)$ AT SOME POINT $X(\theta)$	X or θ	E		$y(X)$ or $r(\theta)$
4	TO EVALUATE THE DERIVATIVE $y'(X)$ AT SOME X	X	f	E	$y'(X)$
5	TO AUTOMATICALLY EVALUATE $y(X)$ OVER A SET OF	X_0	ENTER		
	EQUALLY SPACED POINTS X_i, WHERE $X_i = X_0 + i\Delta$	Δ	f	D	(X_i)
	$(i=0,1,\ldots)$ PRESS $\boxed{R/S}$ TO STOP THE SWEEP				y_i
6	FOR ALL THE FOLLOWING ROUTINES, SPECIFY NUMBER				
	OF SUBINTERVALS IN INTEGRATION $(m = 1, 2, 3, \ldots)$	m	f	C	m
7	TO CALCULATE $\int_{X_1}^{X_2} y(x)\,dx$	X_1	ENTER		X_1
		X_2	A		$\int_{X_1}^{X_2} y(x)\,dx$
8	TO FIND THE AREA BOUNDED BY $y(X)$, THE X AXIS AND THE	X_1	ENTER		X_1
	2 ARBITRARY LIMITS X_1, X_2	X_2	A		AREA
9	TO FIND THE LENGTH OF THE ARC OF $y(X)$ BETWEEN	X_1	ENTER		X_1
	THE ARBITRARY LIMITS X_1, X_2	X_2	B		LENGTH
10	TO FIND THE AREA BOUNDED BY $r(\theta)$ AND 2 ARBITRA-	θ_1	ENTER		θ_1
	RY ANGLES θ_1, θ_2	θ_2	C		AREA
11	TO FIND THE LENGTH OF THE ARC OF $r(\theta)$ BETWEEN	θ_1	ENTER		θ_1
	2 ARBITRARY ANGLES θ_1, θ_2	θ_2	D		LENGTH
12	TO FIND THE VOLUME OF A SOLID OF REVOLUTION GENE-	X_1	ENTER		X_1
	RATED BY $y(X)$ BETWEEN LIMITS X_1, X_2	X_2	f	A	VOLUME
13	TO FIND THE AREA OF THE SURFACE OF REVOLUTION	X_1	ENTER		X_1
	GENERATED BY $y(X)$ BETWEEN LIMITS X_1, X_2	X_2	f	B	AREA
14	FOR ANOTHER CASE, GOTO APPROPIATE STEP				
	WARNINGS ≡ ALL ANGLES ARE ASSUMED TO BE IN				
	RADIANS. THE ANGULAR MODE MUST BE				
	RADIANS ALWAYS.				
	≡ THE # OF SUBINTERVALS IS STORED				
	IN R0 AND IS NOT AFFECTED BY ROUTINES				
	MAY BE CHANGED AT ANY TIME.				
	≡ THE VALUE OF ANY AREA, LENGTH OR				
	VOLUME, AND INTEGRALS IS STORED IN R8				
	≡ THE VALUE OF $y'(x)$ IS STORED IN R3				

Program Listing I

STEP	KEY ENTRY	KEY CODE	COMMENTS
001	*LBL c	32 25 13	STORE # OF SUBINTERVALS
	STO 0	33 00	m IS STORED IN R_0
	RTN	35 22	
	*LBL A	31 25 11	$\int_{x_1}^{x_2} y(x)\,dx$
	1	01	LBL E ≡ 14
	4	04	
	GTO 3	22 03	
	*LBL B	31 25 12	$\int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx$
	0	00	LBL 0 ≡ 0
010	GTO 3	22 03	
	*LBL a	32 25 11	$\int_{x_1}^{x_2} \pi y^2\,dx$
	π	35 73	
	GTO 8	22 08	
	*LBL C	31 25 13	$\int_{\theta_1}^{\theta_2} \frac{1}{2} r^2\,d\theta$
	2	02	$\frac{1}{2}$
	1/x	35 62	
	*LBL 8	31 25 08	STORE $\frac{1}{2}$ OR π IN R_D
	STO D	33 14	
	R↓	35 53	
020	1	01	LBL 1 ≡ 1
	GTO 3	22 03	
	*LBL D	31 25 14	$\int_{\theta_1}^{\theta_2} \sqrt{r^2+r'^2}\,d\theta$
	2	02	LBL 2 ≡ 2
	GTO 3	22 03	
	*LBL b	32 25 12	$\int_{x_1}^{x_2} 2\pi y\sqrt{1+y'^2}\,dx$
	6	06	LBL 6 ≡ 6
	*LBL 3	31 25 03	AUXILIAR LBL
	ST I	35 33	STORE PROPER ADDRESS
	R↓	35 53	
030	*LBL 5	31 25 05	INTEGRATION ROUTINE
	x≷y	35 52	
	STO 6	33 06	STORE LIMITS (a) $\int_a^b f(x)\,dx$
	−	51	
	RCL 0	34 00	$\Delta = \frac{b-a}{m}$ WHERE m= # OF SUBINT.
	STO 1	33 01	
	÷	81	
	STO E	33 15	
	RCL 8	34 08	CLEAR Σ
	STO − 8	33 51 08	
040	*LBL 4	31 25 04	INTEGRATION OF A SUBINTERV
	RCL 6	34 06	x_i
	RCL 6	34 06	$\int_{x_i}^{x_i+\Delta} f(x)\,dx$
	RCL E	34 15	
	STO+ 6	33 61 06	$x_{i+1} \to x_i+\Delta$
	+	61	
	STO 7	33 07	$x_i+\Delta$
	x≷y	35 52	
	STO B	33 12	STORE x_i
	STO − 7	33 51 07	
050	+	61	THE INTEGRAL $\int_a^b f(x)\,dx$ IS COMPUTED AS FOLLOWS:
	STO A	33 11	
	RCL 7	34 07	
	5	05	
	1/x	35 62	
	√	31 54	
	STO C	33 13	

STEP	KEY ENTRY	KEY CODE	COMMENTS
	×	71	A CHANGE OF VARIABLE IS MADE:
	+	61	$x = \frac{b+a}{2} + \frac{b-a}{2}t$
	2	02	$dx = \frac{b-a}{2}dt$
060	÷	81	AND THEN:
	GSB(i)	31 22 24	$\int_{-1}^{1} y(x)dx \simeq$
	STO 9	33 09	$\simeq \frac{5}{6}\left[y\left(\frac{1}{\sqrt{5}}\right)+y\left(-\frac{1}{\sqrt{5}}\right)\right]+$
	RCL A	34 11	$+\frac{1}{6}\left[y(1)+y(-1)\right]$
	RCL 7	34 07	WHICH IS EXACT IF y(x) IS A POLYNOMIAL OF DEGREE 5 OR LESS.
	RCL C	34 13	ALL THIS IS PERFORMED BY STEPS 46 THROUGH 85.
	×	71	
	−	51	
	2	02	
	÷	81	
070	GSB(i)	31 22 24	
	STO+9	33 61 09	
	5	05	
	STO×9	33 71 09	
	RCL B	34 12	
	GSB(i)	31 22 24	
	STO+9	33 61 09	
	RCL 6	34 06	
	GSB(i)	31 22 24	
	STO+9	33 61 09	
080	1	01	
	2	02	
	STO÷9	33 81 09	
	RCL 7	34 07	
	STO×9	33 71 09	DISPLAY $\int_{x_i}^{x_i+\Delta} f(x)\,dx$
	RCL 9	34 09	
	STO+8	33 61 08	Σ
	RCL 1	34 01	ANOTHER SUBINTERVAL
	1	01	
	−	51	
090	STO 1	33 01	
	x≠0	31 61	
	GTO 4	22 04	YES, REPEAT; NO, DISPLAY $\int_a^b f(x)\,dx$
	RCL 8	34 08	
	RTN	35 22	
	*LBL 0	31 25 00	$\sqrt{1+y'^2}$
	GSB e	32 22 15	y'
	1	01	$\sqrt{1+y'^2} \equiv f(x)$
	R→P	32 72	
	RTN	35 22	
100	*LBL 1	31 25 01	πy^2 OR $r^2/2$
	GSB E	31 22 15	y^2 OR r^2
	x²	32 54	
	RCL D	34 14	π OR 1/2
	×	71	πy^2 OR $\frac{r^2}{2} \equiv f(x)$
	RTN	35 22	
	*LBL 2	31 25 02	$\sqrt{r^2+r'^2}$
	STO 2	33 02	
	GSB e	32 22 15	r'
	RCL 2	34 02	
110	GSB E	31 22 15	r
	RCL 3	34 03	r'
	R→P	32 72	$\sqrt{r^2+r'^2} \equiv f(x)$

REGISTERS

0	1	2	3	4	5	6	7	8	9
# OF SUBINTV.	LOOP INDEX	used.	$y'(x)$	used	used	LIMIT OF A SUBINTERVAL	used	INTEGRAL	SUBTOTAL
S0	S1	S2	S3	S4	S5	S6	S7	S8	S9

A	B	C	D	E	I
used ; −Δ	used ; x_i	$\sqrt{\frac{1}{5}}$	used ; $\frac{1}{2}$ OR π	$\frac{b-a}{m}$	ADDRESS

Program Listing II

STEP	KEY ENTRY	KEY CODE	COMMENTS
	RTN	35 22	
	*LBL G	31 25 06	$2\pi y\sqrt{1+y'^2}$
	STO 2	33 02	
	GSB E	31 22 15	y
	STO D	33 14	
	RCL 2	34 02	
	GSB e	32 22 15	y'
120	1	01	$\sqrt{1+y'^2}$
	R→P	32 72	
	RCL D	34 14	
	×	71	
	2	02	
	×	71	$2\pi y\sqrt{1+y'^2} \equiv f(x)$
	π	35 73	
	×	71	
	RTN	35 22	
	*LBL e	32 25 15	DERIVATION ROUTINE
130	STO 5	33 05	
	8	08	
	EEX	43	
	CHS	42	$\Delta = 8 \cdot 10^{-4}$
	4	04	
	STO 4	33 04	
	STO+4	33 61 04	$R_4 \rightarrow 16 \cdot 10^{-4} = 2\Delta$
	STO−5	33 51 05	$x - \Delta x$
	+	61	$x + \Delta x$
	GSB E	31 22 15	f $y(x+\Delta x)$
140	STO 3	33 03	
	RCL 5	34 05	
	GSB E	31 22 15	$y(x-\Delta x)$
	STO−3	33 51 03	
	RCL 4	34 04	
	STO÷3	33 81 03	$y'(x) \simeq \dfrac{y(x+\Delta x)-y(x-\Delta x)}{2\Delta}$
	RCL 3	34 03	
	RTN	35 22	
	*LBL d	32 25 14	AUTOMATIC EVALUATION
	STO A	33 11	STORE Δ
150	X≷Y	35 52	
	*LBL 7	31 25 07	LOOP
	PAUSE	35 72	DISPLAY x_i
	STO B	33 12	
	GSB E	31 22 15	$y(x_i)$
	−x−	31 84	PRINT $y(x_i)$
	RCL B	34 12	
	RCL A	34 11	$x_{i+1} = x_i + \Delta$
	+	61	
	GTO 7	22 07	
160	*LBL E	31 25 15	$y(x)$ EVALUATION
	RTN	35 22	

LABELS

A $x_1 \uparrow x_2 \rightarrow A$	B $x_1 \uparrow x_2 \rightarrow L$	C $\theta_1 \uparrow \theta_2 \rightarrow A$	D $\theta_1 \uparrow \theta_2 \rightarrow L$	E $x \rightarrow y(x)$
a $x_1 \uparrow x_2 \rightarrow V$	b $x_1 \uparrow x_2 \rightarrow S$	c # SUBINTERV	d AUTOMATIC EVALUATION	e $x \rightarrow y'(x)$
0 $\sqrt{1+y'^2}$	1 πy^2 OR $\frac{r^2}{2}$	2 $\sqrt{r^2+r'^2}$	3 auxiliar	4 INTEGRATION OF 1 SUBINTERV.
5 $\int_a^b f(x)dx$	6 $2\pi y\sqrt{1+y'^2}$	7 used	8 used	9

FLAGS

0	
1	
2	
3	

SET STATUS

FLAGS	ON	OFF	TRIG		DISP	
0	☐	■	DEG	☐	FIX	■
1	☐	■	GRAD	☐	SCI	☐
2	☐	■	RAD	■	ENG	☐
3	☐	■			n	4